On some generalizations of the Jacobi identity. II

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Abstract

On the basis of the generalizations of the Jacobi identity found by the author some identities satisfied by the curvature and torsion of a covariant differentiation are derived. A kind of the generalized covariant differentiation is proposed and a method of finding some of satisfied by them identities is given in the correspondence to the curvatures concerned. Possible applications for some physical problems are pointed out.

1. Introduction

In the previous paper [5] we announced a kind of many-point generalizations of the widely used Jacobi identity. The purpose of the present investigation is to give more complete version of [5] including applications in identities satisfied by some derivation operators and the corresponding curvatures.

Let A be an Abelian group and $[\cdot,\cdot] : A \times A \to A$. We say that the operation $[\ ,\ ]$ satisfies the p-th Jacobi identity, $p \geq 2, (cf.[5])$ if for every $A_i \in A, i \in I \neq$

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and $i_1, \dots i_p \in I$, we have

$$\big([A_{i_1}, [A_{i_2}, [\dots, [A_{i_{p-1}}, A_{i_p}]\dots]]]\big)_{[i_1,[i_2,[\dots,[i_{p-1},i_p]\dots]]}$$

$$= p[A_{i_1}, [A_{i_2}, [\dots, [A_{i_{p-1}}, A_{i_p}]\dots]]], \quad p \geq 2, \qquad (1.1)$$

where the (multiple) bracket operation $[\cdot, [\cdot \dots, [\cdot, \cdot]\dots]]$ on the indices is defined as follows:

Let A be Abelian group, the set I be not empty, p and q to be integers, $2 \leq p \leq q, r_1, \dots r_p \in \{1, 2, \dots, q\}, r_a \neq r_b$for $a \neq b, a, b = 1, \dots, p$, and to any $i_1, \dots i_q \in I$ to correspond some $A_{i_1} \in A$. Denoting the opposite element of $A \in A$ by -A, we for $p = 2$ define

$$(A_{i_1 \dots i_q})_{[i_{r_1}, i_{r_2}]} := A_{i_1 \dots i_q} - A_{i_{\tau_2(1)} \dots i_{\tau_2(q)}}$$

where the permutation $\tau_2 : \{1, \dots, q\} \to \{1, \dots, q\}$ is such that $\tau_2(a) := a$ for $a \in \{1, \dots, q\} \backslash \{r_1, r_2\}, \tau_2(r_1) := r_2$, and $\tau_2(r_2) := r_1$.

Let the permutation $\tau_p : \{1, \dots, q\} \to \{1, \dots, q\}$ be defined by $\tau_p(a) := a$ for $a \in \{1, \dots, q\} \backslash \{r_1, \dots, r_p\}, \tau_p(r_1) := r_2, \tau_p(r_2) := r_3, \dots, \tau_p(r_{p-1}) := r_p$and $\tau_p(r_p) := r_1$. Then for $3 \leq p \leq q$ we define

$$A_{i_1} \dots i_q)_{[i_{r_1}, [i_{r_2}, [\dots [i_{r_{p-1}}, i_{r_p}]\dots]}$$

$$:= (A_{i_1 \dots i_q} - A_{i_{\tau_p(1)} \dots i_{\tau_p(q)}})_{[i_{r_2}, [\dots [i_{r_{p-1}}, i_{r_p}]\dots]},$$

where in the first row the square brackets are $p - 1$ and in the second row are $p - 2$.

If (1.1) holds for every $p \leq k$ and $[-A, B] = -[A, B], A, B \in A$, then we call $[\cdot, \cdot]$ antisymmetric of order k. In the case when A is a ring and $[A, B] = [A, B]_- := AB - BA$ is the commutator of A and B the identity (1.1) is valid for every $p \geq 2$. If for [,] we have $[-A, B] = -[A, B]$, then it is antisymmetric of order $k = 2, 3, 4$ iff it satisfies the first k of the following equalities (Sect. 4)

$$\big([A_{i_1}, A_{i_2}]\big)_{<i_1, i_2>} = 0, \qquad (1.2)$$

$$\big([A_{i_1}, [A_{i_2}, A_{i_3}]]\big)_{<i_1, i_2, i_3>} = 0 \quad (\textit{Jacobi identity}), \qquad (1.3)$$

$$\big([A_{i_1}, [A_{i_2}, [A_{i_3}, A_{i_4}]]] + [A_{i_1}, [A_{i_4}, A_{i_3}, A_{i_2}]]]\big)_{<i_1, i_2, i_3, i_4>} = 0, \qquad (1.4)$$

where $i_1, i_2, i_3, i_4 \in I$ and a cyclic summation over the indices included in $<>$ is performed, i.e. $(\dots)_{<i_1 \dots>}$means the sum of all elements of A obtained from $(\dots)$ by a cyclic permutation of the indices $i_1, \dots$.

For $A_{i_1 \dots i_p} \in A,\ p = 2, 3, 4$, we have the identities

$$\big((A_{i_1 \dots i_p} + (-1)^p A_{i_1 i_p i_{p-1} \dots i_2})_{[i_1, [\dots, [i_{p-1}, i_p]\dots]]}\big)_{<i_1 \dots, i_p>} \equiv 0 \qquad (1.5)$$

which reduce to (1.2) – (1.4) respectively, whenever A is a ring and $[A, B] = AB - BA$.

The generalized Jacobi identities, and in particular (1.2) – (1.5), definitely can find applications in the theory of physical systems described by curvature- and/or torsion-depending Hamiltonians (or Lagrangians) [7,8,9,13], especially those containing higher derivatives [9]. This is because the curvature and torsion are simple combinations of commutators of some derivations (see, e.g., (2.1) below or Sec. 4), and - on the other side- the commutators are exactly binary operations for which (some of) the generalized Jacobi identities hold. At the level of classical Hamiltonian mechanics, with or without constraints, there is also a place for possible application of the mentioned identities. In fact, as it is well known, the Poisson bracket is a classical analogue of the quantum commutator is [3]. In our terminology, the Poisson bracket is at least a third order antisymmetric operation, since it is antisymmetric and satisfies the Jacobi identity [3]. Hence, in principle, the generalized Jacobi identities will give a relationship between the curvature and torsion corresponding to some commutators (or, e.g. Poisson brackets). This relationship may turn out to be useful for the physical theories, for instance, for finding conservation laws or first integrals of the equations of motion. Examples of this kind consist in using the (second) Bianchi identity (in the form of the vanishing of the covariant divergence of the Einstein's tensor) for the derivation of conservation laws in general relativity [14], and in using the classical Jacobi identity for generating first integrals of the equations of motion from those already found ones [3].

2. Some Identities for the Covariant Differentiation

Consider a manifold M endowed with an affine connection (covariant differentiation) $\nabla[2, 3]$. Let A, B, C and D be vector fields on M. The curvature $R(A, B)$ and torsion $T(A, B)$ operators are defined by

$$R(A, B) := [\nabla_A, \nabla_B]_- - \nabla_{[A,B]_-}, \qquad (2.1a)$$

$$T(A, B) := \nabla_A B - \nabla_B A - [A, B]_-, \qquad (2.1b)$$

where $[\nabla_A, \nabla_B]_- := \nabla_A \circ \nabla_B - \nabla_B \circ \nabla_A$is the commutator of $\nabla_A$and $\nabla_B$and $[A, B]_- := A \circ B - B \circ A$ is the one of A and B.

Since the algebra of derivations along vector fields is also a ring and $[,]_-$is bilinear, (1.1) is valid for $[,]_-$ (with $A_{i_a} = \nabla_{B_a}, B_a$ being vector fields) for every $p \geq 2$ and, as a consequence of this, the equalities (1.2) – (1.3) are also true for $[,]_-$. Let us see what these equalities mean now.

From (1.2), we find $0 = ([\nabla_A, \nabla_B]_-)_{<A,B>} = (\nabla_A \circ \nabla_B - \nabla_B \circ \nabla_A)_{<A,B>} = (R(A, B) + \nabla_{[A,B]_-})_{<A,B>} = (R(A, B))_{<A,B>}$ or

$$R(A, B) + R(B, A) = 0 \qquad (2.2)$$

which expresses the usual skew-symmetry of the curvature operator [2, 3, 4].

The Jacobi identity (1.3) yields the second Bianchi identity [2,3,4]. In fact, we have

$$0 = ([\nabla_A, [\nabla_B, \nabla_C]_-]_-)_{<A,B,C>}$$

$$
\begin{aligned}
&= (\nabla_A \circ [\nabla_B, \nabla_C]_- - [\nabla_B, \nabla_C]_- \circ \nabla_A)_{<A,B,C>} \\
&= \big((\nabla_A R)(B,C) + \nabla_A \circ \nabla_{[B,C]_-} + R(\nabla_A B, C) \\
&\quad + R(B, \nabla_A C) - \nabla_{[B,C]_-} \circ \nabla_A\big)_{<A,B,C>} \\
&= \big((\nabla_A R)(B,C) + R(A, T(C,B)) + \nabla_{[A,[B,C]_-}\big)_{<A,B,C>}
\end{aligned}
$$

from where, using (2.2) and

$$
(\nabla_{[A,[B,C]_-})_{<A,B,C>} = \nabla_{([A,[B,C]_-} = \nabla_0 = 0,
$$

which is a consequence of (1.3), we get

$$
\big((\nabla_A R)(B,C) + R(T(A,B),C)\big)_{<A,B,C>} = 0. \tag{2.3}
$$

For $p \geq 4$ the equality (1.1) produces new identities satisfied by the curvature and torsion operators (2.1). Of course, these identities are not independent of the known ones [3, 4] but nevertheless they are new. Beneath we shall derive only the first of them corresponding to $p = 4$. In that case (1.4) reduces to

$$
\big([\nabla_A, [\nabla_B, [\nabla_C, \nabla_D]_-]_-]_- + (B \longleftrightarrow D)\big)_{<A,B,C,D>} = 0,
$$

where $+(B \longleftrightarrow D)$ means that we have to add terms obtained from the preceding ones by changing the symbols B and D. Using the definition of $[,]_-$and (2.1a), after some easy manipulation, from the last equality we obtain:

$$
\begin{aligned}
&\big\{(\nabla_A \nabla_B R)(C,D) + (\nabla_A R)\big((B, \nabla_D C - [D,C]_-) + (C, \nabla_B D - \nabla_D B) - (D, \nabla_B C)\big) \\
&\quad + R\big((A, -\nabla_B \nabla_C D + \nabla_C \nabla_D B + \nabla_D [B,C]_- + [B,[C,D]_-]_-) \\
&\quad + (\nabla_A B, \nabla_D C + [C,D]_-) + (\nabla_A C, \nabla_B D)\big) + (B \longleftrightarrow D)\big\}_{<A,B,C,D>} = 0.
\end{aligned}
$$

Writing here explicitly the terms $+(B \longleftrightarrow D)$ and using (2.1), we get after a simple calculations:

$$
\begin{aligned}
&\big\{\big((R(A,B) + \nabla_{[A,B]_-})R\big)(C,D) + (\nabla_A R)(B,[C,D]_-) + (\nabla_A R)([B,C]_-,D) \\
&\quad + R(A, R(C,B)D) + R(A, R(C,D)B) - R(A, T(B,[C,D]_-)) \\
&\quad - R(A, T([B,C]_-, D)) + R(T(A,B),[C,D]_-)\big\}_{<A,B,C,D>} = 0.
\end{aligned} \tag{2.4}
$$

Since A, B, C and D are arbitrary vector fields, the sum of the terms containing commutators in this identity must be zero, i.e. (2.4) reduces to the following two identities:

$$
\big\{\big((R(A,B)R\big)(C,D) + R(R(A,B)C,D) + R(C,R(A,B)D)\big\}_{<A,B,C,D>} = 0 \tag{2.5}
$$

and

$$
\big\{(\nabla_{[A,B]_-} R)(C,D) + (\nabla_A R)(B,[C,D]_-) + (\nabla_A R)([B,C]_-,D)
$$

$$R(A,T(B,[C,D]_-))-R(A,T([B,C]_-,D))$$
$$+R(T(A,B),[C,D]_-)\}_{<A,B,C,D>}=0 \qquad (2.6)$$

It is easy to see that (2.6) is equivalent to

$$\big\{\big((\nabla_{[C,D]_-}R)(A,B)+R(T([C,D],A),B)\big)_{<[C,D],A,B>}\big\}_{<A,B,C,D>}=0, \qquad (2.6')$$

and hence it is evident that (2.6) is a corollary to the second Bianchi identity (2.3). Let us note that if we have grouped the terms into the identity preceding (2.4) in a different way we might get directly (2.5) without the "additional" identity (2.6). Analogous identities, in which only the first derivatives of R appear, may be obtained from (1.1) for $p\geq 5$, but, because of their more complicated algebraic structure, we will not derive them here.

Let us see now how the identities (1.5), which are more "general" then (1.2) – (1.4), can be used for obtaining identities for the torsion and curvature.

For $p=2$ from (1.5) we get

$$0=((\nabla_A B)_{[A,B]})_{<A,B>}=(\nabla_A B-\nabla_B A)_{<A,B>}$$
$$=(T(A,B)+[B,A])_{<A,B>}=(T(A,B))_{<A,B>},$$

i.e.,

$$T(A,B)+T(B,A)=0 \qquad (2.7)$$

which expresses the well known [2, 3] skew-symmetry of the torsion.

For $p=3$ the identity (1,5) reproduces the first Bianchi identity [3, 4]. In fact, it yields

$$0=((\nabla_A\nabla_B C)_{[A,[B,C]]})_{<A,B,C>}$$
$$=(\nabla_A\nabla_B C-\nabla_B\nabla_C A-\nabla_A\nabla_C B+\nabla_C\nabla_B A)_{<A,B,C>}$$
$$=(R(C,B)A+\nabla_A(T(B,C))+T([C,B]_-,A)+[[C,B]_-,A]_-)_{<A,B,C>}$$

which, by means of $(2.2),(2.1a)$ and (1.3), can be rewritten as

$$\big(R(A,B)C-(\nabla_A T)(B,C)-T(T(A,B),C)\big)_{<A,B,C>}=0 \qquad (2.8)$$

As one may expect, (1.5) for $p=4$ generates a new identity for the curvature and torsion of the connection. Since the derivation of this identity is simple but pretty long enough, as that of (2.5), we shall skip the calculations and present here only some of the intermediate results:

$$0=\{(\nabla_A\nabla_B\nabla_C D)_{[A,[B,[C,D]]]}+(B\leftrightarrow D)\}_{<A,B,C,D>}=$$
$$=\cdots=\{(\nabla_A\nabla_B T)(C,D)+(\nabla_A T)(\nabla_B(C,D))+T(\nabla_A\nabla_B(C,D))$$
$$+\nabla_A\nabla_B[C,D]_- -((\nabla_A R)(B,C))(D)-((\nabla_A R)(C,D))(B)$$
$$+(R(\nabla_A(B,C)))(D)-(R(\nabla_A(C,D)))(B)-R(C,D)(\nabla_A B)$$
$$-\nabla_A(\nabla_{[B,C]_-}D+\nabla_{[C,D]_-}B)+\nabla_{[C,D]_-}(\nabla_B A)+(B\longleftrightarrow D)\}_{<A,B,C,D>}=\cdots$$

$$= \{(\nabla_A T)(\nabla_B(C,D) + \nabla_D(C,B))\}_{<A,B,C,D>},$$

where $\nabla_A(B,C) := (\nabla_A B, C) + (B, \nabla_A C)$. After some easy calculations, from the last result, finally, we find

$$\{(R(A,B))(T(C,D)) - (R(A,B)T)(C,D) - T(R(A,B)C,D)$$

$$-T(C,R(A,B)D)\}_{<A,B,C,D>} = 0 \qquad (2.9)$$

A feature of this new identity is that if the connection is curvature or torsion free, then in the both cases it reduces to the trivial one: $0 = 0$.

3. A generalization of the Covariant Differentiation

Let there be given a family $\xi := \{\xi_a : a \in \Lambda \neq\}$ of real vector bundles $\xi_a = (E_a, \pi_a, M), \pi_a : E_a \to M$ over a differentiable manifold M and the dimensions of $\xi_a$and $\xi_b$be equal for every $a,b \in \Lambda$, i.e. $\pi_a^{-1}(x)$ and $\pi_b^{-1}(x), x \in M$ be isomorphic vector spaces. The set of C^ksections of $\xi_a, k \geq 0$, will be denoted by $\mathrm{Sec}^k(\xi_a)$; $\mathrm{Sec}(\xi_a)$ means the set of all sections over ξ_a.

Let us define maps

$$^{a,b}I_{x\to y} : \pi_a^{-1}(x) \to \pi_b^{-1}(y), \quad x,y \in M, \qquad (3.1)$$

which will be called transports; we suppose that:

$$^{b,c}I_{y\to z} \circ^{a,b} I_{x\to y} =^{a,c} I_{x\to z}, \quad a,b,c \in \Lambda, \quad x,y,z \in M \qquad (3.2)$$

and

$$^{a,a}I_{x\to x} = id_{\pi_a^{-1}(x)}. \qquad (3.3)$$

One can easily prove

Proposition 3.1.*A* map (3.1) satisfies (3.2) and (3.3) if and only if there exist a vector space Q of dimension $\dim(\pi_a^{-1}(x))$ and 1:1 maps $^aF_x : \pi_a^{-1}(x) \to Q$ such that

$$^{a,b}I_{x\to y} =^b F_y^{-1} \circ^a F_x. \qquad (3.4)$$

Further we shall need the maps $^{a,b}I_x$:$\mathrm{Sec}(\xi_a) \to$$\mathrm{Sec}(\xi_a)$ defined by

$$(^{a,b}I_x(T))(y) :=^{a,b} I_{x\to y}T(x), \quad T \in Sec(\xi_a),\ x,y \in M. \qquad (3.5)$$

Due to (3.2), we, evidently, have $^{b,c}I_x \circ^{a,b} I_y =^{a,c} I_y$.

On ξ we can define the following generalization of the (linear) covariant differentiation on vector bundles. Let us consider the maps

$$^{a,b}\nabla_V : Sec^1(\xi_a) \to Sec(\xi_b), \qquad (3.6)$$

V being a vector field on M which may be called a *formal connection* on ξ. By definition the maps possess the properties

$$^{a,b}\nabla_V(S+T) =^{a,b} \nabla_V S +^{a,b} \nabla_V T, \qquad (3.7)$$

$$^{a,b}\nabla_{V+W} =^{a,b} \nabla_V +^{a,b} \nabla_W, \qquad (3.8)$$

$$^{a,b}\nabla_{f\cdot V} = f(x) \cdot^{a,b} \nabla_V, \qquad (3.9)$$

$$^{a,b}\nabla_V \circ (f\cdot) =^{a,b} I_x \circ (V(f)\cdot) + f(x) \cdot^{a,b} \nabla_V, \qquad (3.10)$$

where $S, T \in$Sec$^1(\xi_a), V$ and W are vector fields on $M, f : M \to \mathbf{R}, V(f)$ is the action of V on f and $f\cdot$ means (left) multiplication with f.

It is easily seen that the map $\nabla : V \mapsto \nabla_V : \cup_{a\in\Lambda} Sec^1(\xi_a) \to \cup_{a\in\Lambda} Sec(\xi_a)$ defined by $(\nabla_V T)(x) := (^{a,a}\nabla_V T)(x)$ for every $T \in$Sec$^1(\xi_a)$, has the properties $\nabla_V (S+T) = \nabla_V S + \nabla_V T, \nabla_{V+W} = \nabla_V + \nabla_W, \nabla_{f\cdot V} = f \cdot \nabla_V$and $\nabla_V \circ (f\cdot) = V(f) + f \cdot \nabla_V$. Hence ∇ defines a covariant differentiation in every vector bundle ξ_a[3, 4].

The conditions (3.7) – (3.10) imply some restrictions on the used transports. Namely, from (3.8) and (3.10) one immediately derives that $^{a,b}I_{x\to y}((\sigma + \tau)T_x) =^{a,b} I_{x\to y}(\sigma T_x) +^{a,b} I_{x\to y}(\tau T_x)$ for every $\sigma, \tau \in \mathbf{R}$ and $T_x \in \pi_a^{-1}(x)$. In particular this means that the transports must be **Z**-linear. The condition is naturally satisfied if the transports are **R**-linear, i.e., if

$$^{a,b}I_{x\to y}(\sigma S_x + \tau T_x) = \sigma^{a,b} I_{x\to y} S_x + \tau^{a,b} I_{x\to y} T_x, \sigma, \tau \in \mathbf{R} \qquad (3.11)$$

for every $S_x, T_x \in \pi_a^{-1}(x)$, which is assumed hereafter everywhere.

The transports $^{a,b}I_{x\to y}$and the derivatives $^{a,b}\nabla_V$will be called *consistent* if

$$^{a,b}\nabla_V =^{c,b} I_y \circ^{a,c} \nabla_V. \qquad (3.12)$$

From this definition for $c = a$ immediately follows

Proposition 3.2. If $^{a,b}I_{x\to y}$and $^{a,b}\nabla_V$are consistent, then

$$^{a,b}\nabla_V =^{a,b} I_x \circ \nabla_V \qquad (3.13)$$

and, on the opposite, if $\nabla_V : \cup_{\in\Lambda} Sec^1(\xi_a) \to \cup_{a\in\Lambda} Sec(\xi_a)$ preserves the type of the sections, i.e., ∇_V:Sec$^1(\xi_a) \to$Sec(ξ_a) and has the written above properties, then the maps (3.13) satisfy (3.7) – 3.10), are consistent with $^{a,b}I_{x\to y}$and the map Sec$^1(\xi_a) \ni T \mapsto (^{a,a}\nabla_V T)$ coincides with ∇_V.

If the maps $^{a,b}I_{x\to y}$are C^1(with respect to x or y), we can define

$$^{a,b}\nabla_V^I := \lim_{\epsilon\to 0}[\frac{1}{\varepsilon}(^{a,b}I_{x_\epsilon} -^{a,b} I_x)] = \left(\frac{\partial}{\partial\varepsilon}(^{a,b}I_{x_\epsilon})\right)\Big|_{\epsilon=0}, \qquad (3.14)$$

where $x_\epsilon^\alpha = x^\alpha + \epsilon V^\alpha, V = V^\alpha \partial/\partial x^\alpha$in some local basis.

Proposition 3.3. The maps (3.14) satisfy (3.7) – (3.10) and

$$^{b,c}\nabla_V^I \circ^{a,b} I_y \equiv 0, \qquad (3.15)$$

$$^{b,c}\nabla_V^I \circ^{a,b} \nabla_W^I \equiv 0, \qquad (3.16)$$

$$^{b,c}I_y \circ^{a,b} \nabla_V^I =^{a,c} \nabla_V^I, \qquad (3.17)$$

the first of which does not depend on the linearity of $^{a,b}I_{x\to y}$.

Proof. The first statement of the proposition follows directly from (3.14); (3.15) and the consistency condition (3.17) are consequences from (3.14) and ${}^{b,c}I_x \circ^{a,b} I_y =^{a,c} I_y$; (3.16) is a corollary of (3.17) and (3.15).■

At the end of this section we shall write some expressions in local bases.

Let $\{e_i^a, i = 1, \dots, \dim(\xi_a)\}$ and $\{\partial/\partial x^\alpha\}$ be bases in $\mathrm{Sec}(\xi_a)$ and in the bundle tangent to M respectively. We define the transports

$$ {}^{a,b}I_{x\to y} e_i^a(x) =:^{a,b} H^j_{.i}(y,x) e_j^b(y), \qquad (3.18) $$

$$ \big(({}^{a,b}\nabla_{\partial/\partial x^\alpha})(e_i^a)\big)(y) =:^{a,b} \Gamma^j_{.i\alpha}(y,x) e_j^b(y), \qquad (3.19) $$

where the summation is understood to be performed on the repeated on different levels indices within the range of their values. One can easily verify that the consistency condition (3.12) is equivalent to

$$ {}^{a,b}\Gamma^j_{.i\alpha}(z,x) =^{b,c} H^j_{.k}(z,y) {}^{a,c}\Gamma^k_{.i\alpha}(y,x) \qquad (3.20) $$

and if $T = T^i e_i^a$ and $V = V^i \partial/\partial x^\alpha$, then

$$ {}^{a,b}I_{x\to y} T(x) =^{a,b} H^j_{.i}(y,x) T^i(x) e_j^b(y), \qquad (3.21) $$

$$ \begin{aligned} ({}^{a,b}\nabla_V T)(y) = V^\alpha(x)[{}^{a,b}H^j_{.i}(y,x)(\partial T^i/\partial x^\alpha)(x) \\ +{}^{a,b}\Gamma^j_{.i\alpha}(y,x)T^i(x)]e_j^b(y), \qquad (3.22) \end{aligned} $$

$$ ({}^{a,b}\nabla^I_V T)(y) = V^\alpha(x)[{}^{a,b}H^j_{.i}(y,x)(\partial T^i/\partial x^\alpha)(x) +{}^{a,b}H^j_{.i\alpha}(y,x)T^i(x)]e_j^b(y), \qquad (3.23) $$

$$ \begin{aligned} (({}^{b,c}\nabla_V \circ^{a,b} I_y)(T))(z) = V^\alpha(x)\big(-{}^{b,c}H^j_{.i\alpha}(z,x) \\ +{}^{b,c}\Gamma^j_{.i\alpha}(z,x)\big){}^{a,b}H^i_{.k}(x,y)T^k(y)e_j^b(z), \qquad (3.24) \end{aligned} $$

where ($cf.$(3.4))

$$ \begin{aligned} {}^{a,b}H^i_{.j\alpha}(y,x) := \frac{\partial\, {}^{a,b}H^i_{.j}(y,x)}{\partial x^\alpha} \\ = -{}^{a,b}H^j_{.k}(y,x) \frac{\partial\, {}^{b,a}H^k_{.l}(x,y)}{\partial x^\alpha} {}^{a,b}H^l_{.i}(y,x) \qquad (3.26) \end{aligned} $$

From the equality (3.24) follows the evident implication

$$ {}^{b,c}\nabla_V \circ^{a,b} I_y = 0 \Leftrightarrow {}^{b,c}\nabla_V =^{b,c} \nabla^I_V. \qquad (3.27) $$

The purpose of what follows is to find the analogues of curvature and torsion operators for the derivatives ${}^{b,c}\nabla_V$and some satisfied by them identities.

4. Curvatures for the Generalized Covariant Differentiation

Classically [2, 4] the curvature and the torsion of a linear connection arise from the consideration of combinations quadratic with respect to the connection or some vector fields ($cf.$(2.1)), and these combinations are skew-symmetric ($cf.$(2.2)). In our case a number of different such expressions can be formed

which may be divided into two groups. The first of them contains the expressions obtained from

$$ {}^{b,c}\nabla_W \circ^{a,b} \nabla_V \qquad (4.1) $$

by antisymmetrizing it with respect to: V and W; x and y; (x,V) and (y,W); b and c; (b,V) and (c,W); (b,x) and (c,y); (b,x,V) and (c,y,W). The first three antisymmetrizations are formed in a usual way, e.g. that one with respect to V and W gives

$$ {}^{b,c}\nabla_W \circ^{a,b} \nabla_V - {}^{b,c}\nabla_V \circ^{a,b} \nabla_W . $$

The remaining antisymmetrizations have two variants, e.g. that one with respect to (b,x) and (c,y) gives

$$ {}^{b,c}\nabla_W \circ^{a,b} \nabla_V - {}^{b,c}I_z \circ^{c,b} \nabla_W \circ^{a,c} \nabla_V , $$

$$ {}^{b,c}\nabla_W \circ^{a,b} \nabla_V - {}^{b,c}I \circ^{c,b} \nabla_W \circ^{a,c} \nabla_V , $$

where ${}^{b,c}I$:Sec$(\xi_b) \to$Sec(ξ_c) and if $T \in$Sec(ξ_b), then $({}^{b,c}I(T))(x) :=^{b,c} I_{x\to x}(T(x))$. Here the maps ${}^{b,c}I_z$and ${}^{b,c}I$ appear because otherwise the differences are not defined (the different terms in them map one and the same section into sections from, generally, different bundles).

A simple calculation, with the usage of (3.22) and (3.4), shows that if $T \in$Sec$^1(\xi_a)$ and $x,y,t \in M$, then

$$ ({}^{b,c}\nabla_W \circ^{a,b} \nabla_V(T))(t) = W^\alpha(y)V^\beta(x)\Big\{ \Big[{}^{b,c} H^j_{.i\alpha}(t,y)^{a,b}\Gamma^i_{.k\beta}(y,x) $$

$$ {}^{b,c}H^j_{.i}(t,y)\Big(\frac{\partial^{a,b}\Gamma^i_{.k\beta}(z,x)}{\partial z^\alpha}\Big|_{z=y}\Big]T^k(x) + [-{}^{b,c}H^j_{.i\alpha}(t,y) $$

$$ +{}^{b,c}\Gamma^j_{.i\alpha}(t,y)]\delta^\gamma_\beta(({}^{a,b}\nabla_{\partial/\partial x^\gamma})(T))^i(y)\Big\} e^c_j(y). \qquad (4.2) $$

From here we see that any (first or higher) generalized covariant derivative of some C^1section contains only first partial derivatives of that section.

The skewsymmetric expressions of the second class are obtained by antisymmetrizing the expression

$$ {}^{b,c}\nabla_W \circ^{a,b} \nabla_V , \qquad (4.3) $$

where ${}^{a,b}\nabla : V \mapsto^{a,b} \nabla_V$:Sec$^1(\xi_a) \to$Sec$(\xi_b)$ is such that $({}^{a,b}\nabla_V T)(x) := ({}^{a,b}\nabla_V T)(x)$ for $T \in$Sec$^1(\xi_a)$, with respect to V and W, b and c and (b,V) and (c,W).

Applying (4.3) to $T \in$Sec$^2(\xi_a)$ and using twice (3.22), we get

$$ ({}^{b,c}\nabla_W \circ^{a,b} \nabla_V(T))(x) =^{b,c} I_{x\to x}\big((({}^{a,b}\nabla_{W(V^\alpha)\partial/\partial x})T)(x)\big) $$

$$ +W^\beta(x)V^\alpha(x)\{[{}^{b,c}\Gamma^j_{.i\beta}(x,x)^{a,b}\Gamma^i_{.k\alpha}(x,x) + {}^{b,c}H^j_{.i}(x,x) $$

$$ \frac{d^{a,b}\Gamma^i_{.k\alpha}(x,x)}{dx^\beta} - {}^{a,b,c}K^{\mathrm{j}\cdot\gamma}_{.l\beta\alpha}\Gamma^l_{.k\gamma}(x,x)]T^k(x) +^{a,b,c} K^{\mathrm{j}\cdot\gamma}_{.k\beta\alpha} $$

$$(({}^{a,a}\nabla_{\partial/\partial x^\gamma})T)^k(x) + {}^{a,c}H^j_{.k}(x,x)\frac{\partial^2 T^k(x)}{\partial x^\beta \partial x^\alpha}\} e^c_j(x), \qquad (4.4)$$

where

$${}^{a,b,c}K^{\mathbf{j}\cdot\gamma}_{.k\beta\alpha} :={}^{b,c} H^j_{.i}(x,x)(\frac{d^{a,b}H^l_{.k}(x,x)}{dx^\beta}\delta^\gamma_\alpha + {}^{a,b}\Gamma^i_{.k\alpha}(x,x)\delta^\gamma_\beta)$$

$$+{}^{b,c}\Gamma^j_{.i\beta}(x,x)^{a,b}H^i_{.k}(x,x)\delta^\gamma_\alpha, \qquad (4.5)$$

in which $\delta^\gamma_\alpha = 1$ for $\alpha = \gamma$ and $\delta^\gamma_\alpha = 0$ for $\alpha \neq \gamma$. The quantities (4.5) are symmetric with respect to α and β in the following important cases:

$${}^{a,a,a}K^{\mathbf{j}\cdot\gamma}_{.k\alpha\beta} ={}^{a,a} \Gamma^j_{.k\alpha}(x,x)\delta^\gamma_\beta + {}^{a,a}\Gamma^j_{.k\beta}(x,x)\delta^\gamma_\alpha, \qquad (4.6a)$$

$$+{}^{a,b,c}K^{\mathbf{j}\cdot\gamma}_{.k\alpha\beta}|_{\nabla=\nabla^I} ={}^{a,c} H^j_{.k\alpha}(x,x)\delta^\gamma_\beta + {}^{a,c}H^j_{.k\beta}(x,x)\delta^\gamma_\alpha, \qquad (4.6b)$$

where $\nabla = \nabla^I$ means that ${}^{a,b}\nabla_V$ has to be replaced by ${}^{a,b}\nabla^I_V$the action of which on $T \in \mathrm{Sec}^1(\xi_a)$ is $(({}^{a,b}\nabla^I_V)(T))(x) := ({}^{a,b}\nabla^I_V T)(x)$.

If we make the above-pointed antisymmetrizations in (4.2) and (4.4), which are 11 and 5 in number respectively, we shall obtain expressions which are linear with respect to W, V, T and the first generalized covariant derivatives of T(higher derivatives do not occur). The coefficients before the last two quantities are, by definition, the components of the corresponding curvatures. As the consideration of all of these 16 cases is similar, we are going to consider, for brevity, only one of them.

The antisymmetrization of (4.3) with respect to W and V, due to (4.4), gives

$${}^{b,c}\nabla_W \circ^{a,b} \nabla_V - {}^{b,c}\nabla_V \circ^{a,b} \nabla_W - {}^{b,c}I \circ^{a,b} \nabla_{[W,V]_-}$$

$$={}^{a,b,c} R(W,V) + {}^{a,b,c}D(W,V), \qquad (4.7)$$

where

$$({}^{a,b,c}R(W,V)T)(x) ={}^{a,b,c} R^j_{.i\beta\alpha}T^i(x)W^\beta(x)V^\alpha(x)e^c_j(x), \qquad (4.8a)$$

$$({}^{a,b,c}D(W,V)T)(x) ={}^{a,b,c} D^{\mathbf{j}\cdot\gamma}_{.i\beta\alpha}(((\nabla_{\partial/\partial x^\gamma})T)(x))^i W^\beta(x)V^\alpha(x)e^c_j(x), (4.8b)$$

define the first and the second curvature operators (for ${}^{a,b}\nabla$) whose components are:

$${}^{a,b,c}R^j_{.i\beta\alpha} = \big({}^{b,c}H^j_{.k}(x,x)\frac{d}{dx^\beta}({}^{a,b}\Gamma^k_{.i\alpha}(x,x)) + {}^{b,c}\Gamma^j_{.k\beta}(x,x)$$

$${}^{a,b}\Gamma^k_{.i\alpha}(x,x)\big)_{[\alpha,\beta]} - {}^{a,b,c}D^{\mathbf{j}\cdot\gamma}_{.k\beta\alpha}\Gamma^k_{.i\gamma}(x,x) \qquad (4.9a)$$

$${}^{a,b,c}D^{\mathbf{j}\cdot\gamma}_{.i\beta\alpha} := ({}^{a,b,c}K^{\mathbf{j}\cdot\gamma}_{.i\beta\alpha})_{[\alpha,\beta]}. \qquad (4.9b)$$

It is important to note that by virtue of (4.6), we have

$${}^{a,a,a}D^{\mathbf{j}\cdot\gamma}_{.i\beta\alpha} :={}^{a,b,c} D^{\mathbf{j}\cdot\gamma}_{.i\beta\alpha}|_{\nabla=\nabla^I} = 0. \qquad (4.10)$$

The first of these cases, $b = c = a$, corresponds to the usual covariant differentiation in the vector bundles ξ_a. In fact, in this case ${}^{a,a,a}R(W,V)T = R(W,V)T =:^{a} R(W,V)T$, where $R(W,V)$ is given by (2.1a) (see the above definition of $\nabla : V \mapsto \nabla_V$), so ${}^{a}R$ is the usual curvature operator for ∇ in ξ_a, the components of which are obtained from (4.9a) for $c = b = a$, i.e. they are

$$
{}^{a}_{x}R^{j}_{.i\beta\alpha} = \left[\frac{d}{dx^{\beta}} ({}^{a,a}\Gamma^{j}_{.i\alpha}(x,x)) +^{a,a} \Gamma^{j}_{.k\beta}(x,x)^{a,a}\Gamma^{k}_{.i\alpha}(x,x) \right]_{[\alpha,\beta]}
$$

(see (4.10) and (3.3)). Let us note that ${}^{a}_{x}R^{j}_{.i\beta\alpha} = 0$ iff $\nabla = \nabla^{I}$, i.e. iff ${}^{a,a}\Gamma^{k}_{.i\alpha}(x,x) =^{a,a} H^{k}_{.i\alpha}(x,x)$.

Now we are going to describe some of the identities satisfied by the above curvatures.

Since the left-hand side of (4.7) can not be written in terms of commutators of ${}^{a,b}\nabla_V$and ${}^{b,c}\nabla_W$(they simply are not well defined) some of the identities satisfied by the above curvatures may be found on the base of (1.5) or from (1.1) for a special definition of the operation [,]. Below, as examples, we consider the consequences of (1.5) for $p = 2, 3$.

In our case (1.5) for $p = 2$ gives

$$
\begin{aligned}
0 = \big(({}^{b,c}\nabla_W \circ^{a,b} \nabla_V)_{[W,V]}\big)_{<W,V>} &= ({}^{b,c}\nabla_W \circ^{a,b} \nabla_V - {}^{b,c}\nabla_V \circ^{a,b} \nabla_W)_{<W,V>} \\
&= ({}^{b,c}\nabla_W \circ^{a,b} \nabla_V - {}^{b,c}\nabla_V \circ^{a,b} \nabla_W - {}^{b,c}I \circ^{a,b} \nabla_{[W,V]_-})_{<W,V>} \\
&= ({}^{a,b,c}R(W,V) + {}^{a,b,c}D(W,V))_{<W,V>}.
\end{aligned}
$$

Hence, because of (4.8), it follows that

$$
{}^{a,b,c}R(W,V) +^{a,b,c} R(V,W) =^{a,b,c} D(W,V) +^{a,b,c} D(V,W) = 0 \qquad (4.11)
$$

i.e. the skewsymmetry of the curvatures with respect to their vector arguments.

In the case under consideration (1.5) for $p = 3$ gives:

$$
\begin{aligned}
0 &= \big(({}^{c,d}\nabla_A \circ^{b,c} \nabla_B \circ^{a,b} \nabla_C)_{[A,[B,C]]}\big)_{<A,B,C>} \\
&= \big(({}^{c,d}\nabla_A \circ^{b,c} \nabla_B \circ^{a,b} \nabla_C -^{c,d} \nabla_B \circ^{b,c} \nabla_C \circ^{a,b} \nabla_A)_{[B,C]}\big)_{<A,B,C>}.
\end{aligned}
$$

Therefore, taking into account (4.3), we get

$$
\begin{aligned}
&[{}^{c,d}\nabla_A \circ ({}^{a,b,c}R(B,C) +^{a,b,c} D(B,C) +^{b,c} I \circ^{a,b} \nabla_{[B,C]_-}) \\
&-({}^{b,c,d}R(B,C) +^{b,c,d} D(B,C) +^{c,d} I \circ^{b,c} \nabla_{[B,C]_-}) \circ^{a,b} \nabla_A]_{<A,B,C>} = 0. \qquad (4.12)
\end{aligned}
$$

Because of (4.8) and (4.4), this equality is equivalent to two identities which include the curvatures and their first derivatives. These identities are obtained by equating to zero the coefficients before T and $({}^{a,a}\nabla_{\partial/\partial x^{\gamma}})T$ when (4.12) is applied to an arbitrary $T \in \mathrm{Sec}^2(\xi_a)$. The identities pointed out, as well as their derivation, are simple but too long to be presented here.

5. Conclusions

In this work we gave simple examples of application of the generalizations of the classical Jacobi identity announced in [5].

In the case of the covariant differentiation the first and the second of mentioned identities reproduce the known identities satisfied by the curvature and torsion tensors, but beginning with the third one, which is explicitly derived, the proposed here method generates new identities for these tensors.

In Sect. 3 we have proposed a kind of generalization of the covariant differentiation in a family of equidimensional vector bundles over a given differentiable manifold (which formally is supposed to be endowed with a kind of a "transport along that manifold"). Here we wish to point out that a typical example of such families are the tensor bundles of a given rank. For instance, in the notions of Sect. 3, we may set $\xi = \{\xi_1, \xi_2, \xi_3, \xi_4\}$ with $\xi_1 = T(M) \otimes T(M), \xi_2 = T(M) \otimes T^*(M), \xi_3 = T^*(M) \otimes T(M)$ and $\xi_4 = T^*(M) \otimes T^*(M)$, where $T(M)$ is the bundle tangent to M and $T^*(M)$ is its dual. In section 8 we have described how the curvatures of this differentiation operations can be introduced and how on the basis of the generalized Jacobi identities some identities satisfied by them can be obtained. Because, in the general case, there are 16 such identities, we consider in details only two of them. An analogous treatment is still valid for the remaining identities.

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